Lecture 26.

trace free

Recall CR curvature tensor $S_{\alpha\bar\beta\gamma\bar\mu}$ of a s. ψcvx $M \subseteq \mathbb{C}^{n+1}$ at $p_0 \in M$. It was obtained from Moser's normal form at p_0:

$$\operatorname{Im} w = |z|^2 + \sum_{k,l\geq 2} N_{kl}(z,\bar z,\operatorname{Re} w)$$

by $S(Z,\bar Z)$ $\quad S_{\alpha\bar\beta\gamma\bar\mu} Z^\alpha \bar Z^\beta Z^\gamma \bar Z^\mu$ ← summation convention.

$$= N_{22}(Z,\bar Z,0).$$

Here, $Z = Z^\alpha \frac{\partial}{\partial z^\alpha} \in T^{1,0}_{p_0} M$. Going from one normal form $((z,w),N)$ to another $((z',w'),N')$ lead to transformation

$$S'_{\alpha\bar\beta\gamma\bar\mu} = r^2 S_{\gamma\bar\varepsilon\delta\bar\varkappa} U_\alpha^{\ \gamma} \overline{U_\beta^{\ \varepsilon}} U_\nu^{\ \delta} \overline{U_\mu^{\ \varkappa}}$$

where $(z',w') = (rUz + aw, r^2 w) + O(|(z,w)|^2)$,

$U^*U = I$, $r \in \mathbb{R}\setminus\{0\}$, $a \in \mathbb{C}$.

Geometric approach to CR curvature

We fix contact form θ on $(M, \mathcal{V})$ (i.e. real-valued, nonvanishing 1-form, s.t. $\theta^\perp = \mathcal{V} \oplus \overline{\mathcal{V}}$). Recall that Cartan's formula $\Rightarrow$

$$\mathcal{L}^\theta_p(Z,W) := \frac{1}{2i}\theta([Z,\overline{W}]) = -\frac{1}{i}\, d\theta(Z,\overline{W}).$$

Let us choose a local frame $Z_1,\ldots,Z_n$ for $\mathcal{V}$ $\Rightarrow$ Levi form $\rightsquigarrow$ $n\times n$ Hermitian matrix $-g_{\alpha\bar\beta} = \mathcal{L}^\theta(Z_\alpha, Z_\beta) = -\frac{1}{i}\, d\theta(Z_\alpha, \overline{Z}_\beta)$

- Strict pcvxity $\Rightarrow$ $g_{\alpha\bar\beta}$ is definite
  ($\Rightarrow$ θ <u>is</u> contact form)

- Assume θ chosen s.t. $g_{\alpha\bar\beta} > 0$

- Note minus sign, but we shall refer to $g_{\alpha\bar\beta}$ as the Levi form.

Now, choose additional 1-forms $\theta^1, \dots, \theta^n$ s.t. $(\theta, \theta^1, \dots, \theta^n)^\perp = \overline{\mathcal{V}}$ and

$$\theta^\alpha(Z_\beta) = \delta_\beta{}^\alpha .$$

Introduce $\theta^{\bar\alpha} = \overline{\theta^\alpha} \Rightarrow (\theta, \theta^\alpha, \theta^{\bar\alpha})$ is a local coframe. The most general change of coframe preserving these properties and θ is given by

$$(*) \qquad \begin{pmatrix} \tilde\theta \\ \tilde\theta^\alpha \\ \tilde\theta^{\bar\alpha} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ t^\alpha & t_\beta{}^\alpha & 0 \\ \overline{t^\alpha} & 0 & \overline{t_\beta{}^\alpha} \end{pmatrix} \begin{pmatrix} \theta \\ \theta^\beta \\ \theta^{\bar\beta} \end{pmatrix}$$

where (t^α) $\mathbb{C}^n$-valued, $(t_\beta{}^\alpha)$ $GL(n, \mathbb{C})$-valued. The coframe $(\theta, \theta^\alpha, \theta^{\bar\alpha})$ induces a dual frame $(T, Z_\alpha, Z_{\bar\alpha})$, where Z_α are the chosen frame for $\mathcal{V}$ and T a real vector field s.t. $T \notin \mathcal{V} \oplus \overline{\mathcal{V}}$.

The Levi form changes by

$$g_{\alpha\bar\beta} = \hat g_{\nu\bar\mu}\, t_\alpha{}^\nu\, \overline{t_\beta{}^\mu}$$

under change of co/frame (*). If we want to preserve $g_{\alpha\bar\beta}$ (as in Moser's normal form) then $(t_\alpha{}^\beta)$ must be unitary and, for each $p\in M$, we may identify the matrix in (*) at p with the linear part of $\Phi \in \mathrm{Aut}(M_0) \cong SU(1,n+1)$.

<u>Rem</u> Note that fixing the contact form θ is <u>not</u> an invariant for the CR structure ($\theta \to a\theta$ is allowed).

A CR structure with a <u>fixed θ</u> is called a <u>pseudohermitian structure</u> (for reasons we now explain).